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ABSTRACT

The mean-square error of reduced-order linear state estimators for continuous-time linear systems is investigated. Lower and upper bounds on the minimal mean-square error are presented. The bounds are readily computable at each time-point and at steady state from the solutions to the Ricatti and the Lyapunov equations. The usefulness of the error bounds for the analysis and design of reduced-order estimators is illustrated by a practical numerical example.

INTRODUCTION

The need for order reduction in linear state estimation stems from the difficulties associated with the numerical implementation of a full-order optimal estimator for a high-order system. Several suboptimal reduced-order estimator design techniques have been suggested in the literature (e.g., [1]). An optimal mean-square solution for time-invariant linear systems at steady state was suggested by Bernstein and Hyland [2]. Time-varying systems were considered by Sims and Asher [3], who treated the case in which only some of the state variables are of interest. For the more general time-varying case in which reduced-order estimation of the full state vector is desired, there appears to be no tractable optimal solution.

In this paper we first derive a lower bound on the mean-square error of any state estimator having a specified reduced order, for possibly time-varying system. The lower bound is computable for any time-point and, in the case of a time-invariant system, for steady state, using standard algorithms. Its values for all reduced orders are obtained simultaneously and can be readily used to lower-bound the order, which is necessary to achieve a specified performance level. The mean-square error for a given reduced-order estimator can be easily calculated. It can be used to evaluate the estimator quality relative to that of alternative designs, to the lower-bound and to the optimal, full-order estimator. It constitutes an upper bound on the minimal mean-square error, which can be used to eliminate nonoptimal solutions (e.g., local extrema) in an optimization process.

LOWER BOUND ON MEAN-SQUARE ERROR OF A REDUCED-ORDER ESTIMATOR

Consider the system

$$\dot{x}(t) = A(t)x(t) + B(t)w(t) \tag{1a}$$

$$y(t) = C(t)x(t) + v(t) \tag{1b}$$

where $x(t) \in R^n$, $y(t) \in R^m$, $x(0)$ is Gaussian with $E\{x(0)\} = 0$, $E\{x(0)x(0)^T\} = P(0)$, and $w(t)$ and $v(t)$ are mutually uncorrelated Gaussian white-noise processes, uncorrelated with $x(0)$, with $E\{w(t)\} = E\{v(t)\} = 0$, $E\{w(t)w(t)^T\} = Q(t)$, $E\{v(t)v(t)^T\} = R(t)$, and $E\{w(t)v(t)^T\} = S(t)$. An estimator of reduced order $r < n$ for $x(t)$ has the form

$$\dot{q}(t) = F(t)q(t) + G(t)y(t), \qquad q(0) \text{ given} \tag{2}$$

where $q(t) \in R^r$, so that an estimate of $x(t)$ can be obtained by some linear transformation

$$\tilde{x}(t) = H(t)q(t), \qquad \tilde{x}(t) \in R^n \tag{3}$$

It follows that $\tilde{x}(t)$ has, at most, rank r, that is, its covariance matrix has a rank smaller than or equal to r.

Let us denote by $Y(t) = \{y(s), s < t\}$ the σ-algebra of the past observations and by $\hat{x}(t)$ the conditional expectation of $x(t)$, given $Y(t)$. Employing the orthogonality principle, the mean-square error in the estimate $\tilde{x}(t)$ can be written as

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$$
\begin{aligned}
&E\{(x(t) - \tilde{x}(t))^T(x(t) - \tilde{x}(t))\} \\
&\quad = E\{[(x(t) - \hat{x}(t)) + (\hat{x}(t) - \tilde{x}(t))]^T \\
&\qquad \cdot[(x(t) - \hat{x}(t)) + (\hat{x}(t) - \tilde{x}(t))]\} \\
&\quad = E\{(x(t) - \hat{x}(t))^T(x(t) - \hat{x}(t))\} \\
&\qquad + E\{(\hat{x}(t) - \tilde{x}(t))^T(\hat{x}(t) - \tilde{x}(t))\} \qquad (4)
\end{aligned}
$$

Let us denote by $W(t)$ the matrix whose columns are the eigenvectors of $E\{\tilde{x}(t)\tilde{x}(t)^T\}$ arranged in accordance with the corresponding eigenvalues in descending order of magnitude. Let us separate $W(t)$ as $W(t) = [U(t)\ V(t)]$ where $U(t)$ has r columns. Then

$$
W(t)^T\tilde{x}(t) = \begin{bmatrix} \tilde{x}(t)^r \\ 0 \end{bmatrix} \quad \text{a.s.}
$$

where $\tilde{x}(t)^r$ is a vector of dimension r, yielding

$$
\tilde{x}(t) = W(t)\begin{bmatrix} \tilde{x}(t)^r \\ 0 \end{bmatrix} = U(t)\tilde{x}(t)^r \quad \text{a.s.} \qquad (5)
$$

Since $W(t)$ is unitary, we have

$$
W(t)W(t)^T = U(t)U(t)^T + V(t)V(t)^T = I
$$

Hence,

$$
\begin{aligned}
&E\{(\hat{x}(t) - \tilde{x}(t))^T(\hat{x}(t) - \tilde{x}(t))\} \\
&\quad = E\{\hat{x}(t) - U(t)\tilde{x}(t)^r)^T(\hat{x}(t) - U(t)\tilde{x}(t)^r)\} \\
&\quad = \{(\hat{x}(t) - U(t)\tilde{x}(t)^r)^T W(t) \\
&\qquad \cdot W(t)^T(\hat{x}(t) - U(t)\tilde{x}(t)^r)\}
\end{aligned}
$$

yielding

$$
\begin{aligned}
&E\{(\hat{x}(t) - \tilde{x}(t))^T(\hat{x}(t) - \tilde{x}(t))\} \\
&\quad = E\{(\hat{x}(t) - U(t)\tilde{x}(t)^r)^T U(t) \\
&\qquad \cdot U(t)^T(x(t) - U(t)\tilde{x}(t)^r)\} \\
&\qquad + E\{(\hat{x}(t) - U(t)\tilde{x}(t)^r))^T V(t) \\
&\qquad \cdot V(t)^T(\hat{x}(t) - U(t)\tilde{x}(t)^r)\} \qquad (6)
\end{aligned}
$$

which implies

$$
\begin{aligned}
&E\{(\hat{x}(t) - \tilde{x}(t))^T(\hat{x}(t) - \tilde{x}(t))\} \\
&\quad \geq E\{(\hat{x}(t) - U(t)\tilde{x}(t)^r)^T V(t) \\
&\qquad \cdot V(t)^T(\hat{x}(t) - U(t)\tilde{x}(t)^r)\} \\
&\quad = E\{\hat{x}(t)^T V(t)V(t)^T\hat{x}(t)\} \qquad (7)
\end{aligned}
$$

Noting that

$$
\begin{aligned}
V(t)V(t)^T &= I - U(t)U(t)^T \\
&= (I - U(t)U(t)^T)(I - U(t)U(t)^T)
\end{aligned}
$$

we see that

$$
\begin{aligned}
&E\{\hat{x}(t)^T V(t)V(t)^T\hat{x}(t)\} \\
&\quad = E\{\hat{x}(t)^T(I - U(t)U(t)^T)(I - U(t)U(t)^T\hat{x}(t)\} \\
&\quad = \mathrm{tr}\{(I - U(t)U(t)^T)\Pi(t)(I - U(t)U(t)^T)\} \\
&\quad = \mathrm{tr}\{(\Pi(t)^{1/2} - U(t)U(t)^T\Pi(t)^{1/2}) \\
&\qquad \cdot (\Pi(t)^{1/2} - U(t)U(t)^T\Pi(t)^{1/2})\} \qquad (8)
\end{aligned}
$$

where $\Pi(t) = E\{\hat{x}(t)\hat{x}(t)^T\}$.

Let $\lambda_i(t)$, $i = 1,\ldots,n$ denote the eigenvalues of $\Pi(t)$ arranged in descending order of magnitude. Since clearly $U(t)U(t)^T\Pi(t)^{1/2}$ has, at most, rank r, it follows from a well-known result (e.g., [4], p. 63) that

$$
E\{\hat{x}(t)^T V(t)V(t)^T\hat{x}(t)\} \geq \sum_{i=r+1}^{n} \lambda_i(t)
$$

with equality if and only if

$$
\begin{aligned}
&T(t)^T U(t)U(t)^T\Pi(t)^{1/2}T(t) \\
&\quad = \mathrm{diag}(\lambda_1(t)^{1/2},\ldots,\lambda_r(t)^{1/2},0,\ldots,0)
\end{aligned}
$$

It follows that $\varepsilon^2(t)$ is bounded below by

$$
\varepsilon_o^2(t) = \mathrm{tr}\{P(t)\} + \sum_{i=r+1}^{n} \lambda_i(t) \qquad (9)
$$

Since $\varepsilon_o^2(t)$ is a lower bound on the mean-square error of any estimate of $x(t)$ having rank r, it is a lower bound on the mean-square error of any reduced-order state estimator of order r for the system (1).

We note that $P(t)$ satisfies the Riccati equation

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)^T + B(t)Q(t)B(t)^T - [P(t)C(t)^T + B(t)S(t)]R(t)^{-1} \cdot [P(t)C(t)^T + B(t)S(t)]^T \quad (10)$$

and that $\Pi(t)$ satisfies the Lyapunov equation

$$\dot{\Pi}(t) = A(t)\Pi(t) + \Pi(t)A(t)^T + P(t)C(t)^T R(t)^{-1} C(t)P(t) \quad (11)$$

with $\Pi(0) = P(0)$ given. Methods for solving these equations can be found in the literature (e.g., [5]). The lower bound can then be calculated as a function of time by solving (10) and (11). In the time-invariant case, the steady-state value of the lower bound can be obtained by substituting in (9) the steady-state solutions of (10) and (11); that is, the solutions of the algebraic Riccati and Lyapunov equations, respectively.

Once the matrix $\Pi(t)$ is diagonalized, the lower bound is obtained simultaneously for all reduced orders by simply including the corresponding eigenvalues in (9). It can be seen that the lower bound is a monotone decreasing function of the order, which attains its minimal value, $tr\{P(t)\}$, for $r = n$. The error can be used then to determine a lower bound on the order, needed for maintaining the estimation error below a certain level. Specifically, if the maximal acceptable error is $\alpha(t)$, then a lower bound on the order is

$$r_{min} = \min\left\{ r: \sum_{i=r+1}^{n} \lambda_i(t) \le \alpha(t) - tr[P(t)] \right\} \quad (12)$$

This is well illustrated by the closing example.

UPPER BOUND ON MEAN-SQUARE ERROR OF A REDUCED-ORDER ESTIMATOR

Suppose that a reduced-order estimator of order $r < n$ is given by (2). Then the covariance of the estimation error, given by

$$\Omega(t) = E\{[x(t) - H(t)q(t)][x(t) - H(t)q(t)]^T\} \quad (13)$$

can be calculated from

$$\Omega(t) = [I - H(t)]\Gamma(t) \begin{bmatrix} I \\ -H(t) \end{bmatrix} \quad (14)$$

where

$$\Gamma(t) = E\{\underline{x}(t)\underline{x}(t)^T\} \quad (15)$$

with

$$\underline{x}(t) = \begin{bmatrix} x(t) \\ q(t) \end{bmatrix}$$

The latter satisfies the equation

$$\dot{\underline{x}}(t) = \underline{F}(t)\underline{x}(t) + \underline{w}(t) \quad (16)$$

where

$$\underline{F}(t) = \begin{bmatrix} A(t) & 0 \\ G(t)C(t) & F(t) \end{bmatrix} \quad (17)$$

and $\underline{w}(t)$ is a zero mean white-noise process with

$$cov[\underline{w}(t)] = \begin{bmatrix} Q(t) & 0 \\ 0 & G(t)R(t)G(t)^T \end{bmatrix} \quad (18)$$

It follows that $\Gamma(t)$ satisfies the Lyapunov equation

$$\dot{\Gamma}(t) = \underline{F}(t)\Gamma(t) + \Gamma(t)\underline{F}(t)^T + cov[\underline{w}(t)] \quad (19)$$

The mean-square error of the reduced-order estimator is

$$\varepsilon^2(t) = tr\{\Omega(t)\} \quad (20)$$

It should be noted that $\varepsilon^2(t)$ at any time t is computable by standard algorithms for solving (19). When the system and the estimator are time-invariant, $\varepsilon^2(t)$ attains a limit value ε^2 if both A and F have their eigenvalues in the left half-plane.

Minimizing the mean-squared estimation error with respect to the reduced-order estimator parameters is a difficult optimization problem. In order to minimize the error at time t, the optimal matrix functions $\{F(\tau), G(\tau), H(\tau), 0 \le \tau \le t\}$ must be found [3]. For a different time-point, the entire matrix functions must be found anew. Under the assumptions of time-invariance and stability, the problem may be formulated as one of minimizing the steady-state value of the mean-square error ε^2. Even in this form the optimization problem is quite difficult, and its solution requires the simultaneous solution of a set of matrix equations [2].

Clearly, for any specified estimator of the form (2), the mean-square error (20) constitutes an upper bound on the minimal attainable error. It is a useful measure of the quality of the given estimator both in absolute terms and in comparison with alternative reduced-order estimators and with the optimal, full-order one. In the previous section we derived a global lower bound on the mean-square error. A comparison of the specific upper bound, corresponding to a given estimator, to the global lower bound would provide information not only on the domain of the minimal possible error, but also on the quality of the given estimator as a simple, easy-to-obtain alternative to the optimal reduced-order estimator, which is difficult, if not impossible, to derive.

A simple reduced-order estimator for the system (1) is now derived. We first note that the minimum variance estimate $\hat{x}(t)$ of $x(t)$ satisfies the equation

$$\dot{\hat{x}}(t) = [A(t) - K(t)C(t)]\hat{x}(t) + K(t)y(t)$$

where $K(t)$ is the Kalman gain, given by

$$K(t) = [P(t)C(t)^T + B(t)S(t)]R(t)^{-1}$$

It follows that the vector

$$z(t) = T(t)^T\hat{x}(t)$$

satisfies the equation

$$\dot{z}(t) = T(t)^T[A(t) + T(t)\dot{T}(t)^T - K(t)C(t)]T(t)z(t) + T(t)^T K(t)y(t) \tag{21}$$

Suppose that the matrix $\Pi(t)$ has r non-zero eigenvalues. Then

$$\Lambda(t) = E\{z(t)z(t)^T\} = \begin{bmatrix} \lambda_1(t) & & & & \\ & \ddots & & 0 & \\ & & \lambda_r(t) & & \\ & & & 0 & \\ & 0 & & & \ddots \; 0 \end{bmatrix} \tag{22}$$

This means that the values of $z_{r+1}(t),\ldots,z_n(t)$ are zero, with probability 1. These elements can then be eliminated from the vector $z(t)$. Defining

$$q(t) = [I_r \;\; 0]z(t) = [I_r \;\; 0]T(t)^T\hat{x}(t) \tag{23}$$

where I_r is the identify matrix of dimension r, we have

$$\hat{x}(t) = T(t)\begin{bmatrix} I_r \\ 0 \end{bmatrix} q(t) \tag{24}$$

It follows that $q(t)$ satisfies the equation

$$\dot{q}(t) = [I_r \;\; 0]T(t)^T[A(t) + T(t)\dot{T}(t)^T - K(t)C(t)]T(t)\begin{bmatrix} I_r \\ 0 \end{bmatrix} q(t) + [I_r \;\; 0]T(t)^T K(t)y(t) \tag{25}$$

The resulting estimator of order r produces, by means of (24), optimal-state extimates, without degrading the performance of the full-order Kalman filter.

Next, suppose that there are more than r nonzero eigenvalues of $\Pi(t)$. Still, it is desired to reduce the estimator order to r. Eliminating the $n - r$ components of $\hat{z}(t)$ with smallest variances, which are then approximated by zero, the resulting reduced-order estimator is (25), which we now write as

$$\dot{q}(t) = F(t)q(t) + G(t)y(t) \tag{26}$$

with

$$F(t) = [I_r \;\; 0]T(t)^T[A(t) + T(t)\dot{T}(t)^T - K(t)C(t)]T(t)\begin{bmatrix} I_r \\ 0 \end{bmatrix}$$

and

$$G(t) = [I_r \;\; 0]T(t)^T K(t)$$

It can be seen that the difference between the reduced-order estimator (26) and the optimal lies in the elimination of the term

$$[0 \;\; I_{n-r}]T(t)^T[A(t) + T(t)\dot{T}(t)^T - K(t)C(t)T(t)\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}$$

from the dynamics matrix. In the time-invariant case the steady-state value of the bound can be obtained from the corresponding steady-state equations. In this case, the matrix $A - KC$ has all its eigenvalues in the left half-plane and, consequently, so does the matrix $T^T(A - KC)^T T$. This does not guarantee, however, that F has all its eigenvalues in the left half-plane, which can nevertheless be checked numerically for the given system. In the time-varying case the calculation of the upper bound requires integrating (10), (11), and (19), and calculating $T(t)$ at each time-point. The error owing to the above eliminated term may diverge, and the upper bound may not be sufficiently tight to be useful. This, however, can be checked numerically by comparison with the lower bound.

EXAMPLE

The lower and the upper bounds for a seventh-order linear model for a power system, given by

Mahalanbis and Ray [6], are displayed in Fig. 1. It can be seen that for orders greater than 2 the upper and the lower bounds have nearly the same values. This means that for orders greater than 2, the simple reduced-order estimator can be used instead of the optimal reduced-order one, which is difficult to derive. The fact that for orders greater than 3, both bounds are close to the mean-square error of the full-order Kalman filter implies that a simple reduced-order estimator of order 4 can be used instead of the Kalman filter, with an accuracy loss of less than 3% of the optimal error.

CONCLUSION

The design of a reduced-order state estimator which gives a minimal mean-square error is a difficult, if not practically impossible, task. Bounds on the mean-square error are useful for analysis and design purposes. In this paper, we derived a lower bound on the mean-square error of any estimator having a specified reduced order for a given system. The bound is readily computable for any reduced order at any time-point and, for time-invariant systems, at steady state. It is useful for evaluating, by comparison, any given design and for lower-bounding the order needed for a specified accuracy. The latter application offers a considerable reduction in the computation load of any reduced-order estimator design scheme. A specific reduced-order estimator was proposed, from which an upper bound on the minimal mean-square error was calculated. The usefulness of the bounds in reduced-order estimator analysis and design was illustrated by a practical example.

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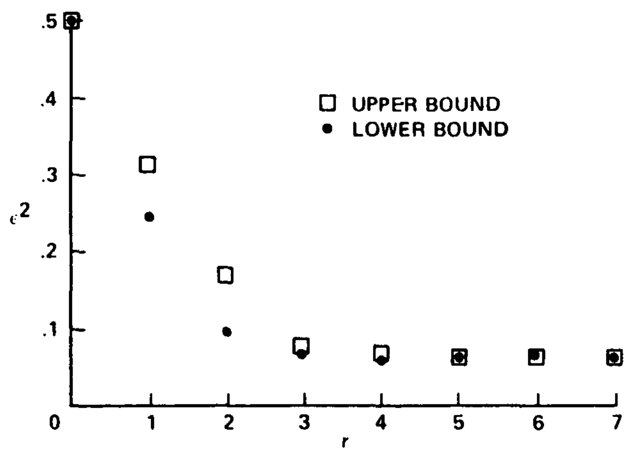

Fig. 1 Lower and upper bounds on minimal mean-square estimation error for the example.

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16. Abstract

The mean-square error of reduced-order linear state estimators for continuous-time linear systems is investigated. Lower and upper bounds on the minimal mean-square error are presented. The bounds are readily computable at each time-point and at steady state from the solutions to the Ricatti and the Lyapunov equations. The usefulness of the error bounds for the analysis and design of reduced-order estimators is illustrated by a practical numerical example.

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